Published by Institute of Physics Publishing for SISSA/ISAS
Received: *February 24, 2004*
Accepted: *April 1, 2004*

Higher conserved charges and integrability for spinning strings in $AdS_5 \times S^5$

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Abstract: We demonstrate the existence of an infinite number of local commuting charges for classical solutions of the string sigma model on $AdS_5 \times S^5$ associated with a certain circular three-spin solution spinning with large angular momenta in three orthogonal directions on the five-sphere. Using the AdS/CFT correspondence we find agreement to one-loop with the tower of conserved higher charges in planar $\mathcal{N} = 4$ super Yang-Mills theory associated with the dual composite single-trace operator in the highest weight representation (J_1, J_2, J_2) of SO(6). The agreement can be explained by the presence of integrability on both sides of the duality.

Keywords: Sigma Models, AdS-CFT and dS-CFT Correspondence, Bethe Ansatz.

 http://jhep.sissa.it/archive/papers/jhep042004002/jhep042004002.pdf

Contents

1. Introduction

The recent realization that the planar one-loop dilatation operator acting on composite gauge invariant operators in $\mathcal{N} = 4$ SU(N) super Yang-Mills theory (SYM) can be mapped to the hamiltonian of an SO(6) spin chain [1], and its supersymmetric extension [2], has uncovered new structures hidden in certain sectors of the gauge theory, most notably integrability. The AdS/CFT correspondence leads one to believe that a similar structure exists on the string theory side, since the scaling dimensions of the operators in the gauge theory are identified with the energies of dual string states. Indeed, in [3, 4] (see also [5]–[8]) it was found that the SO(4, 2) $\times$ SO(6) sigma model (which is known to be classically integrable) evaluated on a particular type of rotating semiclassical string solutions collapses to a one-dimensional Neumann-Rosochatius (NR) system which is known to be integrable. The integrability appears in a rather different way than on the gauge theory side, see [9] for a review. Other evidence of integrability in type IIB string theory on $AdS_5 \times S^5$ was found in [10]–[15]. Furthermore, integrable structures in gauge theories have been found previously in e.g. QCD [16]–[24], but their relation to string theory is not clear.

The energies of the semiclassical rotating string states in the bulk carrying large SO(6) angular momenta J_I, $I = 1, 2, 3$, admit an expansion in the effective coupling constant $\lambda/J^2 = R^4/\alpha'^2 J^2 \ll 1$, where $J = J_1 + J_2 + J_3$, and have been argued to receive no α' corrections in the strict $J \to \infty$ limit, with λ/J^2 kept fixed, [25]–[27]. Furthermore, these states have been conjectured [28] to be dual in the planar limit to holomorphic single-trace operators of the form

$$\mathcal{O} = \mathrm{tr}(\Phi_1 + i\Phi_2)^{J_1}(\Phi_3 + i\Phi_4)^{J_2}(\Phi_5 + i\Phi_6)^{J_3} + \cdots , \tag{1.1}$$

where Φ_A, $A = 1, \ldots, 6$ are the real adjoint scalars in $\mathcal{N} = 4$ SU(N) SYM. The holomorphicity requirement on the operators can be relaxed [29] if e.g. pulsating string solutions are considered [29]–[32]. The dots in (1.1) indicate that to one loop all operators of the same canonical dimension built of only scalars mix among themselves under renormalization, and need to be diagonalized in order to acquire well-defined scaling dimensions. In general this is a highly non-trivial problem when considering operators for which the number of constituent fields J in (1.1) approaches infinity. However, due to the mapping of the one-loop dilatation operator to the hamiltonian of a spin chain the diagonalization procedure can be accomplished with Bethe ansatz techniques [33] which often simplifies in this "thermodynamic limit", when the number of lattice sites on the spin chain approches infinity.

Scaling dimensions of operators of the form given in (1.1) as well as of non-holomorphic operators with insertions of anti-holomorphic "impurities" have been found in several examples using the Bethe ansatz – to one loop in [1, 29] and [34]–[36] and to two loops in [37]. These have been successfully matched to the energies of the dual string states calculated in [25]–[29], [31], and [38]–[41]. There are some indications, however, that this impressive agreement starts to break down at three loops [42, 37].

The integrability on the gauge theory side implies as many integrals of motion as there are constituent elementary fields in the (scalar) operator under consideration, which is equal to the total R-charge J in the case of holomorphic operators. In particular, in the thermodynamic limit an infinite number of conserved charges is expected. In [29] and [43] it was independently realized that the generator of charges in the thermodynamic limit can be identified with the resolvent of the Bethe roots. Focusing on the S^5 part, by applying known Bäcklund transformations for $O(n)$ sigma models [44, 45], these charges were shown to be present also on the string theory side [43] for a certain type of folded and circular two-spin string solutions [3] by constructing the generator of conserved higher charges in the semiclassical limit and extracting the $\mathcal{O}(\lambda)$ contribution. The aim of the present work is to determine the generator of conserved higher charges associated with a circular three-spin solution [28, 40] for which two of the R-charges are equal, say $J_2 = J_3$. This state belongs to a subset of the space of states of the NR system [4] having a constant Lagrange multiplier Λ. For these states one can show that the relation $\sum_I m_I J_I = 0$ must hold for some integers m_I, $I = 1, 2, 3$, which for the (J_1, J_2, J_2) state can be chosen to be $m_1 = 0$, $m_2 = -m_3 = m$. We will find that the generator satisfies a sixth order equation. By restricting to one loop, this sixth order equation factorizes into two cubic equations, one of which is identical to the cubic equation satisfied by the generator on the gauge theory side found in a previous work [29]! The other cubic equation is related to the first one by a discrete transformation.

The paper is organized as follows. In the next section 2 we present a review of the gauge theory calculations, valid to one loop, leading to the infinite set of local commuting charges associated with the operator (1.1) in the SO(6) representation $[0, J_1 - J_2, 2J_2]$,

JHEP04(2004)002

$J_1 > J_2$. In section 3 we summarize the classical three-spin string solution dual to this operator and determine the associated generator of commuting charges by solving a set of Bäcklund equations first considered in the context of type IIB string theory on S^5 in [43]. Some of the details of these calculations are collected in appendices A and B. Finally, in section 4 we give our conclusions.

2. Gauge theory calculations

In this section we review the planar $\mathcal{N} = 4$ SU(N) SYM calculations presented in [29] leading to the generator of conserved charges associated with the operator given in (1.1) having large R-charges, valid to one loop in $\lambda = g_{\mathrm{YM}}^2 N$. There, it was argued that the set of Bethe equations for which $J_1 > J_2 = J_3$ and $J_1 < J_2 = J_3$ need to be considered separately, and reduce respectively to certain types of integral equations found in $O(-1)$ and $O(+1)$ matrix models, see e.g. [46, 47]. Furthermore, it was asserted that the generator in the sector with $J_1 > J_2 = J_3$ coincides with the one in the sector $J_1 < J_2 = J_3$, so for simplicity we shall focus on the former case.

2.1 The one-loop matrix of anomalous dimensions and the Bethe ansatz

We start by presenting the general structure of the planar *one-loop* mixing matrix acting on single-trace composite operators constructed out of J of the three adjoint complex scalars of the SU(N) $\mathcal{N} = 4$ SYM supermultiplet. Restricting to SO(6), the one-loop matrix of anomalous dimensions can be written in the operator form [1]

$$\Gamma = \frac{\lambda}{16\pi^2} \sum_{\ell=1}^{J} \left(K_{\ell,\ell+1} + 2\mathbb{1}_{\ell,\ell+1} - 2P_{\ell,\ell+1} \right), \tag{2.1}$$

acting on a 6^J-dimensional Hilbert space $\mathcal{H} = \bigotimes_{\ell=1}^{J} \mathcal{H}_\ell$, each module carrying the vector representation of SO(6). $K_{\ell,\ell+1}$ and $P_{\ell,\ell+1}$ are trace and permutation operators respectively acting non-trivially only on the modules $\mathcal{H}_\ell$ and $\mathcal{H}_{\ell+1}$. As usual, a given set of operators having the same bare dimensions will generically mix under renormalization and needs to be diagonalized to produce eigenstates of the dilatation operator. We recall that for one-loop renormalization there is no mixing of operators of the form (1.1) containing only scalars with operators containing gluons or fermions. For our purposes, as we analyze operators with large R-charges, the diagonalization procedure can be considerably simplified by recognizing [1] equation (2.1) as the hamiltonian of an SO(6) closed quantum spin chain with nearest-neighbor interactions between the J lattice sites.[1] The spin chain is characterized by the Bethe eigenstates which in the gauge theory correspond to diagonal composite operators possessing well-defined scaling dimensions. Moreover, the Bethe eigenstates are completely specified by their distributions of Bethe roots that solve

[1]The *complete* one-loop dilatation operator of $\mathcal{N} = 4$ SYM [48] is mapped to the hamiltonian of an PSU$(2,2|4) \supset$ SO(6) $\times$ SO(4, 2) super spin chain [2].

the following set of Bethe equations [49, 50]:

$$\begin{aligned}\left(\frac{u_{1,i}+i/2}{u_{1,i}-i/2}\right)^J &= \prod_{j\neq i}^{n_1}\frac{u_{1,i}-u_{1,j}+i}{u_{1,i}-u_{1,j}-i}\prod_{j}^{n_2}\frac{u_{1,i}-u_{2,j}-i/2}{u_{1,i}-u_{2,j}+i/2}\prod_{j}^{n_3}\frac{u_{1,i}-u_{3,j}-i/2}{u_{1,i}-u_{3,j}+i/2}\,,\\ 1 &= \prod_{j\neq i}^{n_2}\frac{u_{2,i}-u_{2,j}+i}{u_{2,i}-u_{2,j}-i}\prod_{j}^{n_1}\frac{u_{2,i}-u_{1,j}-i/2}{u_{2,i}-u_{1,j}+i/2}\,,\\ 1 &= \prod_{j\neq i}^{n_3}\frac{u_{3,i}-u_{3,j}+i}{u_{3,i}-u_{3,j}-i}\prod_{j}^{n_1}\frac{u_{3,i}-u_{1,j}-i/2}{u_{3,i}-u_{1,j}+i/2}\,,\end{aligned} \tag{2.2}$$

where three types of Bethe roots u_I, $I = 1, 2, 3$, are needed, each associated with a simple root of the SO(6) Lie algebra. In (2.2), n_I, $I = 1, 2, 3$ denote the numbers of each type of root for a particular solution and are related to the angular momenta of the composite operator. In particular, for a generic solution carrying three independent angular momenta the relation is $(J_1, J_2, J_3) = (J - n_1, n_1 - n_2 - n_3, n_2 - n_3)$. Note that $n_3 = 0$ for holomorphic operators which saturate the following bound between the canonical dimension Δ_0 and the R-charge (cf. [51, 52]):

$$\Delta_0 \geq J = J_1 + J_2 + J_3\,. \tag{2.3}$$

The SO(6) spin chain is known to be integrable, and we therefore expect the hamiltonian in (2.1) to be only one particular linear combination of a set of J commuting integrals of motion. After diagonalization the eigenvalue of the generator of these commuting charges becomes [33]

$$t(u) = i\log\left(\prod_{k}^{n_1}\frac{u-u_{1,k}+i/2}{u-u_{1,k}-i/2}\right), \tag{2.4}$$

and is accordingly a polynomial in a spectral parameter u:

$$t(u) = \sum_m t_m u^m\,. \tag{2.5}$$

The coefficients t_m are interpreted as higher commuting *local* charges if $t(u)$ is expanded around $u = 0$. In particular, the lowest charges t_0 and t_1 correspond respectively to the total momentum $P = \sum_k p_k$ and the energy eigenvalue E of a quantum eigenstate of the spin chain. Due to the cyclicity of the trace we require $P = 0$. We can now fix the constant of proportionality between the one-loop anomalous dimension and the energy of the spin chain to be

$$\gamma = \frac{\lambda}{8\pi^2}t_1\,, \tag{2.6}$$

which expressed in terms of the u_1 Bethe roots becomes

$$\gamma = \frac{\lambda}{8\pi^2}\sum_{k}^{n_1}\frac{1}{(u_{1,k})^2+1/4}\,, \tag{2.7}$$

obtained by expanding (2.4) to first order in u.

2.2 The gauge dual of the Frolov-Tseytlin string

For the holomorphic operator we are analyzing, i.e. the operator (1.1) with $J_1 > J_2 = J_3$ and with R-charge assignments (J_1, J_2, J_2), or $[0, J_1 - J_2, 2J_2]$ in terms of Dynkin labels, only two types of Bethe roots are needed, as previously mentioned. This highest weight representation actually corresponds to the case of "half-filling" where the number of u_2 roots equals half the number of u_1 roots [29]. In terms of the filling fractions $\alpha \equiv n_1/J$ and $\beta \equiv n_2/J$ this amounts to the condition $\alpha = 2\beta$. By taking the thermodynamic limit, i.e. by letting $J = \sum_I J_I \to \infty$, the Bethe roots accumulate on smooth contours and it is convenient to introduce distribution densities of the Bethe roots, which we normalize as:

$$\int_{\mathcal{C}_+} dv\sigma(v) = \int_{\mathcal{C}_2} dv\rho_2(v) = 1\,. \tag{2.8}$$

The u_1 roots are assumed to be distributed symmetrically around $u = 0$ on two disjoint contours $\mathcal{C}_\pm$ intersecting the real axis, where $\mathcal{C}_-$ is the mirror image of $\mathcal{C}_+$. Furthermore, the u_2 roots are assumed to accumulate on a single curve $\mathcal{C}_2$ on the imaginary axis symmetrically around $u = 0$. Let us rewrite the SO(6) Bethe equations in the thermodynamic limit. By rescaling u by J and taking logarithms, (2.2) reduces to a set of singular integral equations

$$\frac{1}{u} - 2\pi m = \alpha -\!\!\!\!\!\!\int_{\mathcal{C}_+} dv\frac{\sigma(v)}{u-v} + \alpha\int_{\mathcal{C}_+} dv\frac{\sigma(v)}{u+v} - \beta\int_{\mathcal{C}_2} dq\frac{\rho_2(q)}{u-q}\,, \tag{2.9}$$

$$0 = 2\beta -\!\!\!\!\!\!\int_{\mathcal{C}_2} dq\frac{\rho_2(q)}{p-q} - \frac{\alpha}{2}\int_{\mathcal{C}_+} dv\frac{\sigma(v)}{p-v} - \frac{\alpha}{2}\int_{\mathcal{C}_+} dv\frac{\sigma(v)}{p+v}\,, \tag{2.10}$$

where m is an integer enumerating the branch of the logarithm and where a slash through an integral means a Cauchy principal value prescription. In (2.9) and (2.10) u, v and p, q are the u_1 and u_2 roots respectively and we have ignored the u_3 roots. By solving (2.10) for ρ_2 and plugging the result into the normalization condition (2.8), one can show that the u_2 roots distribute themselves on the entire imaginary axis, i.e. $-i\infty < u_2 < i\infty$, and that the following relation holds

$$\int_{-\infty}^{+\infty} dq\frac{\rho_2(iq)}{u-iq} = \int_{\mathcal{C}_+} dv\frac{\sigma(v)}{u+v}\,. \tag{2.11}$$

Thus, eliminating ρ_2 from (2.9), we are left with the following singular integral equation

$$U'(u) = 2 -\!\!\!\!\!\!\int_a^b dv\frac{\sigma(v)}{u-v} - n\int_a^b dv\frac{\sigma(v)}{u+v}\,, \tag{2.12}$$

where $n = -1$. The term on the left hand side of (2.12) can be thought of as the derivative of a (logarithmic) potential $U(u)$ and is for $n = -1$ given by

$$U'(u) = \frac{2}{\alpha}\left(\frac{1}{u} - 2\pi m\right). \tag{2.13}$$

JHEP04(2004)002

The type of integral equations appearing in (2.12) are known to arise as saddle-point equations in $O(n)$ matrix models, see e.g. [46, 47]. By letting $n = 2\cos(\pi p/q)$, where p and q are positive relatively prime integers,[2] the solution, expressed in terms of the resolvent of the eigenvalue density introduced below, is given by a polynomial equation of order q. Hence, for the $O(\pm 1)$ models we expect to find cubic equations.

For a general distribution of Bethe roots on a cut $\mathcal{C}$, the generating function (2.4) turns into

$$t(u) \longrightarrow -\frac{\alpha}{2} W(u)\,, \tag{2.14}$$

in the thermodynamic limit, where we have defined the resolvent of u_1 Bethe roots [29, 43]

$$W(u) \equiv \int_{\mathcal{C}} dv \frac{\sigma(v)}{u-v}\,. \tag{2.15}$$

The resolvent is analytic throughout the complex plane except across the cut $\mathcal{C}$. For a *symmetric* distribution of Bethe roots, as the one we are interested in, we can write $t(u)$ as

$$t(u) = -\frac{\alpha}{2}(W(u) - W(-u))\,, \tag{2.16}$$

where $W(u)$ is the contribution to the resolvent from the cut $\mathcal{C}_+$. Note that for this solution the resolvent of u_2 Bethe roots

$$W_2(p) \equiv \int_{\mathcal{C}_2} dq \frac{\rho_2(q)}{p-q}\,, \tag{2.17}$$

can be expressed in terms of the resolvent of u_1 roots as $W_2(p) = -W(-p)$. In terms of the resolvent, equation (2.12) can be written in the compact form

$$W(u+i0) + W(u-i0) - W(-u) = U'(u)\,. \tag{2.18}$$

We further split the resolvent as $W(u) = W_r(u) + w(u)$, where $w(u)$ solves the homogeneous part of (2.18) and where

$$W_r(u) = \frac{1}{3}(2U'(u) + U'(-u)) = \frac{2}{3\alpha u} - \frac{4\pi m}{\alpha}\,, \tag{2.19}$$

is a particular solution. Utilizing the method in [46, 47], we define the even function

$$r(u) \equiv w^2(u) - w(u)w(-u) + w^2(-u)\,, \tag{2.20}$$

which is regular across the cut and can be determined explicitly to be

$$r(u) = \frac{(4\pi m)^2}{\alpha^2} + \frac{4}{3\alpha^2 u^2}\,, \tag{2.21}$$

by using $W_r(u)$ in (2.19), the regularity of the resolvent (2.15) at $u = 0$ as well as its asymptotic behaviour. Multiplying $r(u)$ by $w(u) - w(-u)$ we find the equation

$$w^3(u) - r(u)w(u) = -w^3(-u) + r(-u)w(-u) \equiv s(u)\,, \tag{2.22}$$

[2]This parametrization is convenient since $O(n)$ matrix models are known to exhibit non-trivial critical behaviour only for $-2 \leq n \leq 2$.

where we have defined the odd function $s(u)$. Since $w(u)$ is regular for $\mathrm{Re}(u) < 0$ and $w(-u)$ is regular for $\mathrm{Re}(u) > 0$, it follows that $s(u)$ is regular everywhere except at $u = 0$. From (2.21) and the behaviour of $w(u)$ at $u = 0$ and $u \to \infty$, $s(u)$ becomes

$$s(u) = \frac{16}{27\alpha^3}\frac{1}{u^3} + 8\frac{(2\pi m)^2}{\alpha^2}\left(1 - \frac{2}{3\alpha}\right)\frac{1}{u}\,. \tag{2.23}$$

Using (2.20) and (2.22) one can verify that the difference $\overline{w} \equiv w(u) - w(-u)$ satisfies the cubic equation

$$\overline{w}^3(u) - r(u)\overline{w}(u) + s(u) = 0\,, \tag{2.24}$$

as previously anticipated. Finally, from the relation

$$\overline{w}(u) = -\frac{4}{3\alpha u} - \frac{2}{\alpha}t(u)\,, \tag{2.25}$$

it is easy to show that $t(u)$ satisfies the cubic equation

$$u^2t^3(u) + 2ut^2(u) + (1 - (2\pi m)^2u^2)t(u) - \alpha(2\pi m)^2u = 0\,. \tag{2.26}$$

This is the main result of this section. Equation (2.26) can easily be solved perturbatively by expanding $t(u)$ around $u = 0$. Hence, using (2.5) as an ansatz the unique odd solution is

$$t(u) = \alpha(2\pi m)^2u + \alpha(1 - 2\alpha)(2\pi m)^4u^3 + \alpha(1 - 6\alpha + 7\alpha^2)(2\pi m)^6u^5 + \cdots\,. \tag{2.27}$$

At lowest order in (2.27) we find a charge proportional to the anomalous dimension γ of the operator under consideration, cf. (2.6). In the thermodynamic limit the anomalous dimension as defined in (2.7) is proportional to the derivative of the resolvent at $u = 0$, so using (2.15) and (2.16) we have

$$\gamma \to \frac{\lambda}{8\pi^2 J}t'(0) = \frac{\lambda m^2 J_2}{J^2}\,. \tag{2.28}$$

Notice that we always have the freedom to linearly combine the charges, a fact that will be made use of when the comparison with the string theory calculation is made.

For the sector in which $J_1 < J_2 = J_3$ an integral equation similar to (2.12) is derived, but with $n = +1$ and a different potential. In [29] it was argued that this sector is the analytic continuation of the $J_1 > J_2 = J_3$ sector past an apparent critical point at $J_2 = 4J_1$, so the generators resulting from the $O(\pm 1)$ models coincide. From a string theoretic perspective, to which we turn to in the coming section, this is natural since no critical behavior is seen at $J_2 = 4J_1$.

3. String theory calculations

The goal of this section is to find the generating function of the higher commuting charges from a string theoretic point of view, following the analysis in [43]. Actually, we will find the charges to all orders in λ (at the semiclassical level), so the one-loop contribution needs to be extracted in order to compare with the gauge theory result. We start by reviewing some of the features of the three-spin string solution with R-charge assignments (J_1, J_2, J_2) on S^5 first considered in [28] and further developed in [40]. From the Bäcklund transformations [43]–[45], for the $O(6)$ sigma model we find a one-parameter family of string solutions given as a function of a certain spectral parameter by treating the three-spin solution as the trial solution. We find that the redefinition of charges on the string side is exactly the same as in [43]. However, using the freedom to linearly combine the charges, we find it more convenient to work with an "improved" spectral parameter introduced below at the cost of slightly complicating the equations but making the comparison with the gauge theory calculations more direct.

3.1 The Frolov-Tseytlin string solution

We begin with summarizing the part of the (J_1, J_2, J_2) solution relevant for the present analysis and refer the reader to [28, 40] for details. We consider a closed string at the center of AdS_5 ($\rho = 0$ in global coordinates) rotating in three orthogonal planes on the five-sphere. The AdS_5 time coordinate is denoted by $t = X_0(\tau, \sigma)$ and we work with the complexified five-sphere embedding coordinates $Z_I(\tau, \sigma) \equiv X_{2I-1}(\tau, \sigma) + iX_{2I}(\tau, \sigma)$, $I = 1, 2, 3$, where (τ, σ) parametrize the string world-sheet. Using conformal gauge, the equations of motion and the conformal gauge constraints for strings on the five-sphere are given in world-sheet light-cone coordinates $\xi = (\tau + \sigma)/2$, $\eta = (\tau - \sigma)/2$ by

$$Z_{\xi\eta} + \mathrm{Re}(Z_\xi \cdot \bar{Z}_\eta) Z = 0\,, \qquad Z \cdot \bar{Z} = 1\,, \qquad Z_\xi \cdot \bar{Z}_\xi = t_\xi^2\,, \qquad Z_\eta \cdot \bar{Z}_\eta = t_\eta^2\,, \tag{3.1}$$

where $Z_\xi \equiv \partial_\xi Z$ and $Z_\eta \equiv \partial_\eta Z$. The closed string periodicity condition is also required. The (J_1, J_2, J_2) solution is given by:

$$t = \kappa\tau\,, Z_I(\tau, \sigma) = u_I(\sigma) \exp(i w_I \tau)\,, \qquad I = 1, 2, 3\,, \tag{3.2}$$

where two of the angular velocities are equal, say $w_2 = w_3 = w$ and $w_1 = \nu$, and where $u_I(\sigma)$ parametrizes the unit 2-sphere

$$\begin{aligned} u_1(\sigma) &= \cos\gamma_0\,, \\ u_2(\sigma) &= \sin\gamma_0 \cos m\sigma\,, \\ u_3(\sigma) &= \sin\gamma_0 \sin m\sigma\,. \end{aligned} \tag{3.3}$$

Here κ, γ_0 and m are constants. From the equations of motion and the conformal gauge constraints in (3.1) a set of relations is obtained between the parameters given in the solution (3.2) and (3.3)

$$w^2 = \nu^2 + m^2\,, \qquad \kappa^2 = \nu^2 + 2m^2 \sin^2\gamma_0\,, \qquad m \in \mathbb{Z}\,. \tag{3.4}$$

In the following we will treat κ and ν as independent parameters. Their expansions in powers of the effective string coupling $\lambda' = \lambda/J^2$ are given in appendix A. The space-time energy of the string is expressed in terms of κ as

$$E = \sqrt{\lambda}\kappa = J + \frac{\lambda m^2 J_2}{J^2} + \cdots\,, \tag{3.5}$$

which via the AdS/CFT correspondence is identified with the scaling dimension of the corresponding operator (1.1) in the gauge theory. To lowest order we recognize the canonical dimension counting the number of constituent fields and to first order in λ/J^2 the one-loop anomalous dimension, cf. (2.28).

3.2 The Bäcklund transformations

It is well-known that $O(n)$ sigma models are classically integrable, see e.g. [44, 45], which in particular implies the existence of an infinite number of conserved charges. One way of constructing *local* conservation laws is to use parametric Bäcklund transformations. Given a "trial" solution $Z(\tau,\sigma)$ to the classical equations of motion in (3.1) a one-parameter family of solutions $Z^{(\gamma)}(\tau,\sigma)$ with $\gamma \in \mathbb{R}$ also solving (3.1) can be obtained by requiring a certain set of Bäcklund equations be satisfied. This "dressed" solution can then be used to construct the generator of conserved charges.

At the cost of making the equations look a little bit more complicated one can make the correspondence between the gauge theory and the string theory calculations more explicit by expanding the dressed solution not in the spectral parameter γ usually considered, see e.g. [43]–[45], but in an "improved" spectral parameter

$$\mu = \frac{\gamma}{1+\gamma^2}\,, \qquad |\mu| \leq \frac{1}{2}\,, \tag{3.6}$$

also considered in [43].[3] Inverting this quadratic equation in γ yields

$$\gamma = \frac{1}{2\mu}(1-\sqrt{1-4\mu^2})\,, \tag{3.7}$$

where the requirement that $\gamma = 0$ should correspond to $\mu = 0$ singles out one solution and consequently excludes the region $|\gamma| > 1$. The Bäcklund equations determining the dressed string solution $Z_I^{(\mu)}$ then take the form [45]

$$\begin{aligned}
\left(1-\sqrt{1-4\mu^2}\right)\left(Z_I^{(\mu)} + Z_I\right)_\xi &= +\mathrm{Re}\left(Z^{(\mu)}\cdot \bar{Z}_\xi\right)\left(Z_I^{(\mu)} - Z_I\right),\\
\frac{4\mu^2}{1-\sqrt{1-4\mu^2}}\left(Z_I^{(\mu)} - Z_I\right)_\eta &= -\mathrm{Re}\left(Z^{(\mu)}\cdot \bar{Z}_\eta\right)\left(Z_I^{(\mu)} + Z_I\right),
\end{aligned} \tag{3.8}$$

together with the normalization and "initial" conditions

$$Z^{(\mu)}\cdot \bar{Z}^{(\mu)} = 1\,, \qquad Z_I^{(\mu)}(\tau,\sigma)|_{\mu=0} = Z_I(\tau,\sigma)\,, \tag{3.9}$$

$$\mathrm{Re}(Z^{(\mu)}\cdot \bar{Z}) = \sqrt{1-4\mu^2}\,, \tag{3.10}$$

where $Z_I(\tau,\sigma)$ is any trial solution solving the classical equations of motion (3.1).

Knowing the dressed solution $Z^{(\mu)}$ the generator of local commuting charges $\mathcal{E}(\mu) = \sum_m \mathcal{E}_m \mu^m$ can be found from[4]

$$\mathcal{E}(\mu) = \frac{1}{4\pi\mu(1-\sqrt{1-4\mu^2})}\int d\sigma \mathrm{Re}\left(2\mu^2 Z^{(\mu)}\cdot \bar{Z}_\xi + (1-2\mu^2-\sqrt{1-4\mu^2})Z^{(\mu)}\cdot \bar{Z}_\eta\right), \tag{3.11}$$

whose conservation law can be derived from (3.8)–(3.10).

[3] In [43] the relation (3.6) was presented as $1-4\mu^2 = (1-\gamma^2)^2/(1+\gamma^2)^2$.

[4] $\mathcal{E}(\mu)$ is related to $\mathcal{E}(\gamma)$ in [43] by the redefinition $\mathcal{E}(\mu) = \mathcal{E}(\gamma)/\gamma^2$.

JHEP04(2004)002

3.3 Finding the generating function

Next, we determine the generating function $\mathcal{E}(\mu)$ of higher charges evaluated on the classical solution (3.2) to an arbitrary order in the spectral parameter μ and to all orders in $\lambda = R^4/\alpha'^2$ in the strict $J \to \infty$ limit in which quantum corrections are suppressed. To achieve this we make a certain ansatz of the dressed solution, first considered in [43], plug it into (3.11) and then use the Bäcklund transformations (3.8) to show that $\mathcal{E}(\mu)$ satisfies a sixth order equation (recall that $t(u)$ satisfy the cubic equation (2.26)). In order to match this generator with the generator $t(u)$ found in the gauge theory (2.27) we extract the one-loop contribution by a certain rescaling of $\mathcal{E}(\mu)$ and by letting $\mathcal{J} = J/\sqrt{\lambda}$ approach infinity, a procedure very reminiscent of that in [43]. For details we refer the reader to appendix B.

Inspired by [43], we make the following ansatz for the dressed solution

$$Z_I^{(\mu)}(\tau,\sigma) \equiv X_{2I-1}^{(\mu)} + iX_{2I}^{(\mu)} = r_I(\sigma,\mu)\exp\left(i\alpha_I(\mu)\right)\exp(iw_I\tau)\,, \tag{3.12}$$

where

$$\begin{aligned} r_1(\sigma,\mu) &= \cos\gamma_0\,, \\ r_2(\sigma,\mu) &= \sin\gamma_0\cos\left(m\sigma + \theta(\mu)\right), \\ r_3(\sigma,\mu) &= \sin\gamma_0\sin\left(m\sigma + \theta(\mu)\right). \end{aligned} \tag{3.13}$$

Here the shift function $\theta(\mu)$ and the phases $\alpha_I(\mu)$ are functions of only the spectral parameter μ and vanish for $\mu = 0$, i.e. $Z^{(\mu)}$ reduces to Z for $\mu = 0$. The one-parameter family of solutions in (3.12) solves the equations of motion (or equivalently the equations of motion of the NR system [4] with constant Λ and $m_2 = -m_3 = m$, $m_1 = 0$) and the conformal gauge constraints in (3.1) for arbitrary μ provided that the relations in (3.4) hold.

Substituting the ansatz (3.12) into (3.11) and performing some elementary integrals the generating function becomes

$$\mathcal{E}(\mu) = \frac{1}{2\mu(1-\sqrt{1-4\mu^2})}\mathrm{Re}\left(2\mu^2 Z^{(\mu)}\cdot\bar{Z}_\xi + (1-2\mu^2-\sqrt{1-4\mu^2})Z^{(\mu)}\cdot\bar{Z}_\eta\right)\Big|_{\sigma=0}. \tag{3.14}$$

By further plugging in the ansatz (3.12) into the Bäcklund equations (3.8) yields an overdetermined system of equations out of which four are independent. This determines the functions $\alpha_I(\mu)$, $I = 1,2,3$ and $\theta(\mu)$ completely. Solving (3.8) for $\mathrm{Re}(Z^{(\mu)}\cdot\bar{Z}_\xi)$ and $\mathrm{Re}(Z^{(\mu)}\cdot\bar{Z}_\eta)$, substituting the result into (3.14), the generator takes the very simple form

$$\mathcal{E}(\mu) = \mu\frac{2\nu}{\sin\alpha_1}\,. \tag{3.15}$$

Now from (3.8) and (3.10) $\sin\alpha_1$ can be shown to satisfy a sixth order equation which consequently determines an equation for the generating function

$$4\nu^2\mu^4\vartheta^2 + \mathcal{E}^2\mu^2\left(4m^4\left(1-4\mu^2\right) + \vartheta\left(4\nu^2-\vartheta\right)\right) + \mathcal{E}^4\left(\kappa^2 - 8m^2\mu^2\right) - \mathcal{E}^6 = 0\,, \tag{3.16}$$

where for notational simplicity we have defined

$$\vartheta \equiv 2m^2 + \nu^2 - \kappa^2 = 2m^2\cos^2\gamma_0\,. \tag{3.17}$$

Let us consider some limiting cases. When $J_1 = 0$ the string reaches its maximal size ($\gamma_0 = \pi/2$) implying that the function ϑ vanishes and (3.16) simplifies to a quartic equation. This corresponds to a string carrying the representation $[J_2, 0, J_2]$[5] and was analyzed on the gauge theory side in [34]. Considering instead the limit $J_2 = J_3 = 0$ we recover the BMN state [25] in the representation $[0, J_1, 0]$ in which the string is point-like. For this solution $\kappa = \nu$ and $m = 0$, so also here ϑ vanishes. Hence, (3.16) shows that the generating function $\mathcal{E}(\mu) \to \kappa$ and no higher charges are present.

Using the expansions of κ and ν in powers of λ/J^2 given in appendix A, the equation (3.16) can be solved perturbatively in (even) powers of μ

$$\mathcal{E}(\mu) = \sum_{m=0}^{\infty} \mathcal{E}_{2m} \mu^{2m} \,. \tag{3.18}$$

The resulting expressions for the for the first few charges $\mathcal{E}_m = \mathcal{E}_m(\kappa, \nu)$ evaluated on our three-spin solution are given in appendix B. Expanding in λ/J^2 to first order we find

$$\begin{aligned}
\sqrt{\lambda}\mathcal{E}_0 &= J + \frac{\lambda m^2 J_2}{J^2} + \cdots \,, \\
\sqrt{\lambda}\mathcal{E}_2 &= -\frac{2^3 \lambda m^2 J_2}{J^2} + \cdots \,, \\
\sqrt{\lambda}\mathcal{E}_4 &= +\frac{2^5 \lambda^2 m^4 J_2}{J^5}(J - 4J_2) + \cdots \,, \\
\sqrt{\lambda}\mathcal{E}_6 &= -\frac{2^7 \lambda^3 m^6 J_2}{J^8}(J^2 - 12 J_2 J + 28 J_2^2) + \cdots \,, \\
&\ \vdots
\end{aligned} \tag{3.19}$$

These charges satisfy "BMN scaling" [25, 43] where the m-th charge scales as $\mathcal{J}^{1-m}$. We observe that the charges t_{2m+1}, $m = 0, 1, 2, \ldots$, found in (2.27) in the gauge theory appear as the *leading order* terms in $\mathcal{E}_{2m+2}$. The one-loop generating function $\mathcal{Q}(u)$ is then easily extracted from $\mathcal{E}(\mu)$ by the following limiting procedure[6]

$$u\mathcal{Q}(u) \equiv \lim_{\mathcal{J}\to\infty} \left(\frac{\mathcal{E}(\mu)}{\mathcal{J}} - 1 \right) , \tag{3.20}$$

where we have identified the spectral parameter of the gauge theory generating function

$$u^2 = -\frac{\mu^2}{\pi^2 \mathcal{J}^2} \,. \tag{3.21}$$

In taking this limit we have lost information about the zeroth order charge, proportional to the energy. To provide ourselves with a "proof" of the one-loop correspondence at a functional level we take the limit (3.20) directly in (3.16), and find that the left hand

[5]This (unstable) circular two-spin solution [28] is a limit of the more general two-spin solution of the NR system with the shape of a *bent* circle considered in [3, 43].

[6]This closely resembles the limiting procedure in [43] for the two-spin case, the main difference being that $\mathcal{Q}(u)$ here is rescaled by a factor of u. Also, $\mathcal{E}(\mu)$ given here differs from $\widetilde{\mathcal{E}}(\mu)$ in [43] by a factor of μ^2.

JHEP04(2004)002

side of the sixth order equation factorizes into two cubic factors:

$$\begin{aligned}&\left[\mathcal{Q}(u\mathcal{Q}+1)^2-(2\pi m)^2u(u\mathcal{Q}+\alpha)\right]\times\\ \times\ &\left[u^3\mathcal{Q}^3+4u^2\mathcal{Q}^2+u\mathcal{Q}(5-(2\pi m)^2u^2)+(2\pi m)^2u^2(\alpha-2)+2\right]=0\,,\end{aligned}\tag{3.22}$$

where we have used the "filling fraction" $\alpha=2\mathcal{J}_2/\mathcal{J}$ previously defined. The first square bracket we recognize as the left hand side of the cubic equation (2.26), since $\mathcal{Q}(u)$ can be identified with $t(u)$! The second factor in (3.22) is related to the first by the discrete transformation

$$\mathcal{Q}(u)\longrightarrow-\frac{2}{u}-\mathcal{Q}(u)\,,\tag{3.23}$$

and appears to yield no new information.

We conclude this section by presenting an explicit expression for the one-loop generating function appropiate for computing the charges perturbatively. From (3.11) and (3.20) an expansion around $\mu=0$ yields

$$u\mathcal{Q}(u)=\lim_{\mathcal{J}\to\infty}\frac{1}{4\pi\mu\mathcal{J}}\int d\sigma\mathrm{Re}\left(\left(Z^{(\mu)}\cdot\bar{Z}_\xi\right)-2\mu\mathcal{J}+\left(\left(Z^{(\mu)}\cdot(\bar{Z}_\eta-\bar{Z}_\xi\right)\right)\mathcal{P}(\mu)\right)\,,\tag{3.24}$$

where $\mathcal{P}(\mu)$ is the polynomial

$$\mathcal{P}(\mu)=\mu^2+\mu^4+2\mu^6+5\mu^8+14\mu^{10}+42\mu^{12}+\cdots\,.\tag{3.25}$$

The expansions of $\mathrm{Re}(Z^{(\mu)}\cdot\bar{Z}_\xi)$ and $\mathrm{Re}(Z^{(\mu)}\cdot\bar{Z}_\eta)$ in powers of μ and $1/\mathcal{J}$ are given in appendix B for the (J_1,J_2,J_2) solution.

4. Conclusions

The main objective of this study was to gain further clues about the relation between the integrability on the two sides of the planar AdS/CFT duality by considering a specific example, using the method initiated in [43]. Recall that integrability appears very differently on the two sides: on the gauge theory side we consider a quantum integrable spin chain, whereas on the string theory side we consider a classical string sigma model. By dressing a specific three-spin solution in type IIB string theory we found a one-parameter family of string solutions $Z^{(\mu)}$ and further derived a function $\mathcal{E}(\mu)$ generating an infinite tower of charges in involution. To first order in the expansion parameter $1/\mathcal{J}$ these were successfully matched onto the gauge theory result found in [29].

We have gained confidence that the limit in (3.20) is the way to extract the one-loop contribution from the string theory generator of higher charges, cf. [43].[7] This strongly suggests that there should be a more general prescription relating the generating functions on the two sides of the duality in the semiclassical (thermodynamic) limit, not specific to any particular solution. Eq. (3.24) should be regarded as an explicit mapping relating the higher charges on the two sides of the duality, given that a string solution is known. Furthermore, a map relating the Bäcklund equations and the Bethe equations in the semiclassical limit is suggestive.

[7] Note the similarity of the result in the present paper with the ones in [43]. In particular, if the generator is expanded in powers of γ instead of $\mu=\gamma/(1+\gamma^2)$, $\widehat{\mathcal{E}}(\gamma)=\sum_{m=2}^{\infty}\widehat{\mathcal{E}}_m\gamma^m$, the *same* order by order redefinitions of the charges as in [43] are required to produce the improved charges.

At the time of this writing there is evidence that perturbative integrability is valid in the planar limit to two loops in $\mathcal{N}=4$ SYM, see for instance [53, 37]. Further conjectures of higher-loop integrability on the gauge theory side have been put forward in e.g. [54, 55, 35], assuming perturbative BMN scaling. On the string theory side we know that classical integrability of the $\mathrm{SO}(4,2)\times\mathrm{SO}(6)$ sigma model holds to all orders. With this in mind, it would be of interest to generalize the results on the gauge theory side to higher orders. To two loops we expect that operators containing only scalars start to mix with operators containing fermions and field strengths. One way to proceed is to calculate the generating function of conserved higher charges using an analog of the integrable Inozemtsev long range spin chain [37].

Acknowledgments

I am grateful to J. Minahan and K. Zarembo for discussions and comments on the manuscript and to L. Freyhult and M. Smedbäck for discussions.

A. Expansions of κ and ν

For the (J_1, J_2, J_2) solution three components of the SO(6) angular momentum tensor J_{AB}, $A, B = 1, \ldots, 6$ are non-zero, corresponding to rotations on the five-sphere in three orthogonal planes:

$$J_1 = J_{12}\,, \qquad J_2 = J_{34}\,, \qquad J_3 = J_{56}\,. \tag{A.1}$$

We recall [40] that the only restriction on the R-charges in this solution comes from the stability requirement $J_2 + J_3 \leq (4m-1)J_1/(2m-1)^2$, where m is the winding number. Evaluating these components using the solution (3.2) and the definitions

$$J_I \equiv \frac{i\sqrt{\lambda}}{4\pi}\int d\sigma\left(Z_I\partial_\tau \bar{Z}_I - \bar{Z}_I\partial_\tau Z_I\right), \qquad \text{(no sum)} \tag{A.2}$$

one easily infers the following relations between the space-time energy of the string $E = \sqrt{\lambda}\kappa$ and the "charge" $\mathcal{V} = \sqrt{\lambda}\nu$ [28, 40]

$$E^2 = \mathcal{V}^2\left(1 + \frac{2\lambda m^2}{\mathcal{V}^2}\left(1 - \frac{J_1}{\mathcal{V}}\right)\right), \tag{A.3}$$

$$(\mathcal{V} - J_1)^2\left(1 + \frac{\lambda m^2}{\mathcal{V}^2}\right) = 4J_2^2\,, \tag{A.4}$$

where E has been eliminated in the last equation using (A.3). Solving these equations as an expansion in powers of $\lambda' = R^4/\alpha'^2 J^2$ and the rescaled charges $\mathcal{J} = J/\sqrt{\lambda}$ and $\mathcal{J}_2 = J_2/\sqrt{\lambda}$ yield

$$\begin{aligned}
\kappa &= \mathcal{J} + \lambda' m^2 \mathcal{J}_2 - \frac{\lambda'^2 m^4 \mathcal{J}_2}{4} + \frac{\lambda'^3 m^6 \mathcal{J}_2}{8}\left(1 - \frac{4\mathcal{J}_2}{\mathcal{J}} + \frac{8\mathcal{J}_2^2}{\mathcal{J}^2}\right) + \cdots\,, \\
\nu &= \mathcal{J} - \lambda' m^2 \mathcal{J}_2 + \frac{\lambda'^2 m^4 \mathcal{J}_2}{4}\left(3 - \frac{8\mathcal{J}_2}{\mathcal{J}}\right) + \frac{\lambda'^3 m^6}{8}\left(\frac{36\mathcal{J}_2}{\mathcal{J}} - 5 - \frac{56\mathcal{J}_2^2}{\mathcal{J}^2}\right) + \cdots\,.
\end{aligned} \tag{A.5}$$

B. Details of the Bäcklund solution and the generating function

Let us collect the main ingredients needed to derive equations (3.15) and (3.16), valid for the (J_1, J_2, J_2) solution. Putting $m\sigma = 0, \pi/2$ in (3.10) we find the conditions that $\alpha_2 = \alpha_3$ and that

$$\cos^2\gamma_0 \cos\alpha_1 + \sin^2\gamma_0 \cos\theta \cos\alpha_2 = \sqrt{1 - 4\mu^2}\,. \tag{B.1}$$

The information that we need from the six components of the complex Bäcklund equations in (3.8) is most easily extracted by taking the real parts and putting $\sigma = 0$. After some algebra we get the following relations between the μ-dependent functions $\theta(\mu)$ and $\alpha_I(\mu)$

$$\begin{aligned} \cos\theta &= \frac{\nu}{\sqrt{m^2\sin^2\alpha_1 + \nu^2}}\,, \qquad & \sin\theta &= \frac{m\sin\alpha_1}{\sqrt{m^2\sin^2\alpha_1 + \nu^2}}\,, \\ w^2\cos^2\theta &= \nu^2 + m^2\cos^2\alpha_2\,, \qquad & w\sin\theta &= m\sin\alpha_2\,. \end{aligned} \tag{B.2}$$

The imaginary parts can be shown to give rise to no additional information. To derive (3.15) we need the integrals

$$\begin{aligned} \int_0^{2\pi} d\sigma \mathrm{Re}(Z^{(\mu)}\cdot\bar{Z}_\xi) &= 2\pi\mathrm{Re}(Z^{(\mu)}\cdot\bar{Z}_\xi)|_{\sigma=0}\,, \\ \int_0^{2\pi} d\sigma \mathrm{Re}(Z^{(\mu)}\cdot\bar{Z}_\eta) &= 2\pi\mathrm{Re}(Z^{(\mu)}\cdot\bar{Z}_\eta)|_{\sigma=0}\,, \end{aligned} \tag{B.3}$$

as well as the expressions

$$\begin{aligned} \mathrm{Re}(Z^{(\mu)}\cdot\bar{Z}_\xi)|_{\sigma=0} &= \nu\cos^2\gamma_0\sin\alpha_1 + \sin^2\gamma_0(m\cos\alpha_2\sin\theta + w\sin\alpha_2\cos\theta)\,, \\ \mathrm{Re}(Z^{(\mu)}\cdot\bar{Z}_\eta)|_{\sigma=0} &= \nu\cos^2\gamma_0\sin\alpha_1 + \sin^2\gamma_0(-m\cos\alpha_2\sin\theta + w\sin\alpha_2\cos\theta)\,, \end{aligned} \tag{B.4}$$

which have the following expansions in powers of $1/\mathcal{J}$ and μ:

$$\begin{aligned} \mathrm{Re}(Z^{(\mu)}\cdot\bar{Z}_\xi)|_{\sigma=0} &= \left(2\mathcal{J} + \frac{2m^2\mathcal{J}_2}{\mathcal{J}^2} + \cdots\right)\mu + \\ &\quad + \left(-\frac{8m^2\mathcal{J}_2}{\mathcal{J}^2} + \frac{12m^4\mathcal{J}_2}{\mathcal{J}^4}\left(\frac{4\mathcal{J}_2}{\mathcal{J}} - 1\right) + \cdots\right)\mu^3 + \cdots\,, \\ \mathrm{Re}(Z^{(\mu)}\cdot\bar{Z}_\eta)|_{\sigma=0} &= \left(2\mathcal{J} - \frac{6m^2\mathcal{J}_2}{\mathcal{J}^2} + \cdots\right)\mu + \\ &\quad + \left(\frac{8m^2\mathcal{J}_2}{\mathcal{J}^2} + \frac{4m^4\mathcal{J}_2}{\mathcal{J}^4}\left(3 - \frac{20\mathcal{J}_2}{\mathcal{J}}\right) + \cdots\right)\mu^3 + \cdots\,. \end{aligned} \tag{B.5}$$

Eliminating $\cos\alpha_2$ and $\cos\theta$ from (B.1) gives an equation for α_1. In particular, by squaring (B.1), one can eliminate the $\cos\alpha_1$ term and express all α_1-dependence in powers of $\sin\alpha_1$:

$$\begin{aligned} m^4\cos^4\gamma_0\sin^6\alpha_1 + m^2\sin^4\alpha_1\left(m^2(1-4\mu^2) + 2\nu^2\cos^2\gamma_0 - m^2\cos^4\gamma_0\right) + \\ +\nu^2\sin^2\alpha_1\left(2m^2(1-4\mu^2) + \nu^2 - 2m^2\cos^2\gamma_0\right) - 4\mu^2\nu^4 = 0\,. \end{aligned} \tag{B.6}$$

Finally, we get (3.16) by expressing $\cos\gamma_0$ in terms of κ and ν using (3.4) and expressing $\sin\alpha_1$ in terms of $\mathcal{E}(\mu)$ using (3.15). Note that to first order in μ (B.6) gives that $\sin\alpha_1 = 2\mu\nu/\kappa + \mathcal{O}(\mu^2)$ which from (3.15) implies that $\lim_{\mu\to 0}\mathcal{E}(\mu) = \kappa$ as it should.

Let us also summarize the first few charges evaluated on our three-spin solution, as functions of κ and ν. These can be determined either by solving the Bäcklund equations (3.8) perturbatively as in [43, section 3.3], a procedure with a rapidly increasing degree of difficulty, or by solving (3.16) perturbatively. Either way the lowest-lying charges become

$$
\begin{aligned}
\mathcal{E}_0 &= \kappa\,,\\
\mathcal{E}_2 &= \frac{(\nu^2-\kappa^2)}{2\kappa^3}(\kappa^2-3\nu^2+4m^2)\,,\\
\mathcal{E}_4 &= \frac{(\nu^2-\kappa^2)}{8\kappa^7}\Big((\kappa^2-\nu^2)(5\kappa^4+14\kappa^2\nu^2+45\nu^4)+\\
&\qquad\qquad + m^2(8\kappa^4+48\kappa^2\nu^2-120\nu^4)+m^4(16\kappa^2-80\nu^2)\Big)\,,\\
\mathcal{E}_6 &= \frac{(\nu^2-\kappa^2)}{16\kappa^{11}}\Big(7(\nu-\kappa)^2(\nu+\kappa)^2(\kappa^2+3\nu^2)(3\kappa^4+2\nu^2\kappa^2+27\nu^4)+\\
&\qquad\qquad + 4m^2(\nu^2-\kappa^2)(567\nu^6-133\kappa^2\nu^4-43\kappa^4\nu^2-7\kappa^6)+\\
&\qquad\qquad + m^4(240\kappa^4\nu^2-2800\kappa^2\nu^4+3024\nu^6+48\kappa^6)+\\
&\qquad\qquad + m^6(64\kappa^4-896\kappa^2\nu^2+1344\nu^4)\Big)\,,
\end{aligned}
\tag{B.7}
$$

where the factorization $(\nu^2-\kappa^2)$ for the charges $\mathcal{E}_m$, $m\geq 2$ ensures that there are no higher charges for the BPS protected point-like string state [25], for which $\nu=\kappa$.

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JHEP04(2004)002

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